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Function-valued Korovkin systems without quasiconcavity and set-valued Korovkin systems without convexity

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Abstract

In this paper, we show how to construct Korovkin systems in spaces of continuous mappings whose values are (possibly non-convex) sets or more generally (possibly non-quasiconcave) upper semicontinuous functions. The Korovkin system is constructed from a given Korovkin system of real functions. Furthermore, we show that any Korovkin system in the quasiconcave case, augmented by all constant functions, is a Korovkin system for the general case.

Keywords: Korovkin system; Linear positive operator; Set-valued mapping; Upper semicontinuous function; Non-convex set

1. Introduction

The main topic of this paper is Korovkin-type approximation of continuous mappings whose values are in a certain space of upper semicontinuous (u.s.c.) functions. The original theorem by Korovkin [10,11] is the following. Let $\mathscr{C}([0,1],\mathbf{R})$ be the class of all real continuous functions on $[0,1]$, and let $\{T_n\}_n$ be a sequence of

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linear positive operators from $\mathscr{C}([0,1],\mathbf{R})$ to $\mathscr{C}([0,1],\mathbf{R})$. Then, in order to conclude that $T_n f \to f$ in the supremum norm for all $f \in \mathscr{C}([0,1],\mathbf{R})$, it is enough to prove convergence for the three functions $1, x, x^2$. This is, no doubt, a most remarkable result which has generated a lot of research since its inception. One now says that $1, x, x^2$ is a Korovkin system for the space $\mathscr{C}([0,1],\mathbf{R})$. The reader is referred to the monograph [1] for an exposition of the theory and many of its applications.

The theory is essentially concerned with the study of Korovkin systems in contexts more general than $\mathscr{C}([0,1],\mathbf{R})$. In the framework of real continuous functions, several characterizations of Korovkin systems are known, see e.g. [1,3]. Therefore, it is natural to investigate whether real Korovkin systems can be used to construct Korovkin systems for functions with values in more general spaces, such as locally convex spaces (more generally, locally convex cones [9]) or hyperspaces (spaces of sets). There are a number of papers where this problem is considered, e.g. [5,8,15].

A serious limitation of the existing Korovkin theorems for set-valued mappings since the pioneering work of Vitale ([19], see also e.g. [4,8,9,15]) is the requirement that the mappings be convex valued. This was so because those results were obtained using linear space or special convex cone techniques. However, a hyperspace containing one non-convex set can never be a convex cone in the senses of Keimel and Roth ('locally convex cone' [9]), of Prolla ('metric convex cone' [12,13]), or of Jonasson ('lattice cone' [6]), let alone be embeddable into a linear space.

In [18], López-Díaz and the author have proven a Korovkin-type approximation theorem for mappings whose values are in a space of u.s.c. functions. That space is endowed with a special structure that makes it a natural generalization of the corresponding space of compact (possibly *non-convex*) sets. Our result shows that convexity assumptions can be disposed of.

This paper continues that research. By Theorem 3.1 in [18], certain *special* real Korovkin systems are useful for constructing set- and function-valued Korovkin systems. However, since its proof makes an essential use of the special properties, it remained unclear whether *every* real Korovkin system could similarly provide a function-valued Korovkin system. In this paper, that question will be answered in the affirmative (Theorem 3). Our main tools for that purpose will be a generalization of the mentioned theorem (Theorem 1) and a recent embedding theorem for the space of u.s.c. functions under the hypothesis of quasiconcavity, due to the author [16,17].

This contributes to the understanding of Korovkin systems in the non-linear space of u.s.c. functions. As a consequence of Theorem 3, we also show the relationship between Korovkin systems with and Korovkin systems without the hypothesis of quasiconcavity (or convexity, in the set-valued case): every Korovkin system $\mathscr{J}$ for quasiconcave functions can be made into a Korovkin system for general u.s.c. functions by adding to it all (non-quasiconcave) constant functions (Theorem 4). It is shown by an example that $\mathscr{J}$ alone need not have the Korovkin property in the general case.

2. Preliminaries

Let $(\mathbf{E}, ||\cdot||)$ be a Banach space. Let $\mathscr{K}$ denote the class of all non-empty compact subsets of $\mathbf{E}$, let $\mathscr{K}_{\mathrm{c}}$ be the subclass of convex elements and let B denote the unit ball of $\mathbf{E}$.

In $\mathscr{K}$, Minkowski addition and product by a scalar are defined by

$$K + L = \{x + y \mid x \in K,\ y \in L\}, \quad \lambda K = \{\lambda x \mid x \in K\}.$$

The Hausdorff metric is defined by

$$d_{\mathrm{H}}(K, L) = \inf\{\varepsilon > 0 \mid K \subset L + \varepsilon B,\ L \subset K + \varepsilon B\}.$$

The *norm* or *magnitude* of $K \in \mathscr{K}$ is

$$||K|| = d_{\mathrm{H}}(K, \{0\}) = \max_{x \in K} ||x||.$$

The indicator function and convex hull of K are denoted by I_K and $\mathrm{co}\, K$, respectively. The norm of a $\mathscr{K}$-valued mapping is defined by $||X||(x) = ||X(x)||$.

The following function spaces will be considered. We will denote by $\mathscr{F}$ (respectively, $\mathscr{F}_{\mathrm{c}}$) the class of all real upper semicontinuous functions from $\mathbf{E}$ to $[0, 1]$ whose maximum is 1 and whose upper level sets are in $\mathscr{K}$ (respectively, $\mathscr{K}_{\mathrm{c}}$). The upper level sets of $A \in \mathscr{F}$ will be denoted by $A_\alpha = \{x \in \mathbf{E} \mid A(x) \geqslant \alpha\}$, for $\alpha \in (0, 1]$, whereas A_0 will denote the closed support of A. Observe that elements of $\mathscr{F}_{\mathrm{c}}$ are quasiconcave, that is, $A(\lambda x + (1 - \lambda)y) \geqslant \min\{A(x), A(y)\}$ for all $\lambda \in [0, 1]$ and $x, y \in \mathbf{E}$.

The quasiconcave envelope of $A \in \mathscr{F}$ will be denoted by qco A (in [18], the notation co A was used). It is determined by the identities $(\mathrm{qco}\, A)_\alpha = \mathrm{co}\, A_\alpha$ for $\alpha \in (0, 1]$, and it is an element of $\mathscr{F}_{\mathrm{c}}$.

Operations in $\mathscr{F}$ have a levelwise characterization, that is,

$$(\lambda A)_\alpha = \lambda A_\alpha \quad \text{and} \quad (A + C)_\alpha = A_\alpha + C_\alpha.$$

Since one of the distributive laws does not hold in general, $\mathscr{F}$ is not a convex cone in any of the senses mentioned in the introduction.

$\mathscr{F}$ can be endowed with a metric d_∞ defined by

$$d_\infty(A, C) = \sup_{\alpha \in [0,1]} d_{\mathrm{H}}(A_\alpha, C_\alpha).$$

$(\mathscr{F}, d_\infty)$ is complete [14], but not separable [7].

It is customary to denote by $A \subset C$ the usual pointwise order $A \leqslant C$, since $A \subset C$ if, and only if $A_\alpha \subset C_\alpha$ for all α. Observe that the operations in $\mathscr{F}$ do *not* correspond to the pointwise ones. Recall that $\mathscr{F}$ should be considered a superspace of $\mathscr{K}$: the mapping $K \mapsto I_K$ embeds linearly $\mathscr{K}$ into $\mathscr{F}$.

From now on Ω will denote a compact Hausdorff topological space. If (M, d) is a metric space, $\mathscr{C}(\Omega, M)$ will denote the set of continuous mappings from Ω to M and analogously $\mathscr{B}(\Omega, M)$ will denote the set of bounded mappings from Ω to M. The identity mapping of a space will be denoted by **id**.

In $\mathscr{C}(\Omega,\mathscr{F})$ we consider the D_∞ metric defined by

$$D_\infty(X,Y) = \sup_{x\in\Omega} d_\infty(X(x),Y(x)).$$

For any $X\in\mathscr{B}(\Omega,\mathscr{F})$ we set $\|X\|_{\mathscr{C}} = \sup_{x\in\Omega}\|X_0\|(x)$.

A mapping $T:\mathscr{C}(\Omega,\mathscr{F})\to\mathscr{C}(\Omega,\mathscr{F})$ will be said to be *linear* if $T(aX+bY) = aT(X)+bT(Y)$, whenever $a,b\geqslant 0$ and $X,Y\in\mathscr{C}(\Omega,\mathscr{F})$, and it will be called *positive* if $T(X)\subset T(Y)$, whenever $X\subset Y$. It will be called an $\mathscr{F}$*-operator* [18] if there is some mapping $\hat{T}:\mathscr{F}\to\mathscr{F}$ such that $T(X)=\hat{T}\circ X$. Observe that $\mathscr{F}$-operators need not be linear or positive.

These definitions specialize to $\mathscr{K}$-valued mappings. Then we will naturally speak of $\mathscr{K}$-operators instead.

Notice that, beyond the formal resemblance, linearity as we have defined it has striking differences from linearity with respect to the pointwise operations. For instance, the mapping $A\mapsto A^2$ is linear in our sense. More generally, the same is true of $A\mapsto\varphi\circ A$ for any increasing bijection φ from $[0,1]$ to $[0,1]$.

Let $\mathscr{D},\mathscr{D}'$ be function spaces. A subset $\mathscr{H}\subset\mathscr{D}$ will be called a *Korovkin system* for an operator T, for mappings from $\mathscr{D}$ to $\mathscr{D}'$, if for every sequence $T_n:\mathscr{D}\to\mathscr{D}'$ of linear positive operators, convergence $T_nf\to Tf$ in $\mathscr{H}$ entails convergence in all of $\mathscr{D}$. We will naturally say that $\mathscr{H}$ is a Korovkin system in $\mathscr{D}$ when $\mathscr{D}=\mathscr{D}'$. Constant functions will be denoted by their only value. An expression like $\mathscr{H}x$ is understood as the set of all $h\cdot x$, h ranging over $\mathscr{H}$.

3. Korovkin systems

In this section we will develop the main results of the paper. The first one is an improvement of Theorem 3.1 in [18] which will be our main tool. Part (iii) of the theorem and uniform convergence over a parameter set Λ will not be used in the sequel. However, we prefer to state the theorem in the same generality as [18, Theorem 3.1].

Theorem 1. *Let Ω be a compact Hausdorff topological space, and let $\phi:\Omega^2\to\mathbf{R}$ be a continuous mapping such that $\phi(x,x)=0$ for all $x\in\Omega$ and $\phi(x,y)>0$ for all $x,y\in\Omega$ with $x\neq y$. Let $u\in\mathscr{F}_c$ be such that $aI_B\subset u$ for some $a>0$. Let $L_{n,\lambda}:\mathscr{C}(\Omega,\mathscr{F})\to\mathscr{B}(\Omega,\mathscr{F})$, $n\in\mathbf{N}$, $\lambda\in\Lambda$ (Λ being an index set), be positive linear operators, such that there exists $n_0\in\mathbf{N}$ with*

$$\sup_{\lambda\in\Lambda,n\geqslant n_0}\|L_{n,\lambda}(u)\|_{\mathscr{C}}<\infty.$$

Let $T:\mathscr{C}(\Omega,\mathscr{F})\to\mathscr{C}(\Omega,\mathscr{F})$ be an $\mathscr{F}$-operator.

Then, the following conditions are equivalent:

(i) $D_\infty(L_{n,\lambda}(X),T(X))\to 0$ *uniformly in* $\lambda\in\Lambda$ *for all* $X\in\mathscr{C}(\mathscr{F})$,

(ii) $D_\infty(L_{n,\lambda}(\phi(x,\cdot)u),T(\phi(x,\cdot)u))\to 0$ *uniformly in* $\lambda\in\Lambda$ *for all* $x\in\Omega$,
$D_\infty(L_{n,\lambda}(A),T(A))\to 0$ *uniformly in* $\lambda\in\Lambda$ *for all* $A\in\mathscr{F}$,

(iii) $\sup_{x\in\Omega} ||(L_{n,\lambda}(\phi(x,\cdot)I_B))_0||(x)\to 0$ *uniformly in* $\lambda\in\Lambda$,
$D_\infty(L_{n\lambda}(A), T(A))\to 0$ *uniformly in* $\lambda\in\Lambda$ *for all* $A\in\mathscr{F}$.

Proof. The proof is similar to that of Theorem 3.1 in [18]. Let us just underline the differences: $\mathbf{E}$ substitutes for $\mathbf{R}^d$ and a general u replaces I_B. The former is actually immaterial to the method of the proof.

As to the latter, define

$$d_u(A, C) = \inf\{\varepsilon>0 \mid A\subset C + \varepsilon u, C\subset A + \varepsilon u\}.$$

It is readily seen that d_u is a metric. Actually, d_∞ and d_u are uniformly equivalent, since

$$A\subset C + \varepsilon u \Rightarrow A\subset C + \varepsilon||u_0||I_B \quad \text{and} \quad A\subset C + \varepsilon I_B \Rightarrow A\subset C + \varepsilon a^{-1}u.$$

Define accordingly

$$D_u(X, Y) = \sup_{x\in\Omega} d_u(X(x), Y(x)) \quad \text{and} \quad ||X||^u_{\mathscr{C}} = D_u(X, I_{\{0\}}).$$

By inspection of the proof of Theorem 3.1 in [18], we conclude that just two inequalities must hold for the method to be applicable to D_u.

The first one is $D_u(fu, gu)\leqslant ||f-g||_\infty \cdot ||u_0||$ for $f, g\in\mathscr{C}(\Omega, \mathbf{R}^+)$, which is easy to prove. (In [18] one had $D_\infty(fI_B, gI_B) = ||f-g||_\infty$, but the former is enough.)

The second one is that $||T(X)||^u_{\mathscr{C}}\leqslant ||T(u)||^u_{\mathscr{C}} \cdot ||X||^u_{\mathscr{C}}$ for any linear positive operator T and $X\in\mathscr{C}(\Omega, \mathscr{F})$. In order to prove this, notice that

$$X\subset ||X||^u_{\mathscr{C}} \cdot u \Rightarrow T(X)\subset ||X||^u_{\mathscr{C}} \cdot T(u)\subset ||X||^u_{\mathscr{C}} \cdot ||T(u)||^u_{\mathscr{C}} \cdot u$$

and

$$I_{\{0\}}\subset X + ||X||^u_{\mathscr{C}} \cdot u \Rightarrow I_{\{0\}} = T(I_{\{0\}})\subset T(X) + ||X||^u_{\mathscr{C}} \cdot T(u)\subset T(X) + ||X||^u_{\mathscr{C}} \cdot ||T(u)||^u_{\mathscr{C}} \cdot u,$$

whence

$$||T(X)||^u_{\mathscr{C}} = D_u(X, I_{\{0\}})\leqslant ||X||^u_{\mathscr{C}} \cdot ||T(u)||^u_{\mathscr{C}}.$$

With those inequalities to hand, one can adapt the aforementioned proof so as to show the equivalence of

(i) $L_{n,\lambda}(X)\to T(X)$ for all $X\in\mathscr{C}(\Omega, \mathscr{F})$,
(ii) $L_{n,\lambda}(\phi(x,\cdot)u)\to T(\phi(x,\cdot)u)$ for all $x\in\Omega$, and $L_{n,\lambda}(A)\to T(A)$ for all $A\in\mathscr{F}$,
(iii) $\sup_{x\in\Omega} ||(L_{n,\lambda}(\phi(x,\cdot)u))_0||(x)\to 0$, and $L_{n,\lambda}(A)\to T(A)$ for all $A\in\mathscr{F}$,

where convergence is uniform in $\lambda\in\Lambda$ and in the D_u sense, which is of course the same as D_∞ convergence by the uniform equivalence of the metrics. □

The other main tool we will use is the following embedding of $\mathscr{F}_c$ into a space of continuous functions [16, Corolario 1.1.6; 17, Theorem 6].

Lemma 2. *There exist a compact Hausdorff space Γ and a mapping $h : \mathscr{F}_c \to \mathscr{C}(\Gamma, \mathbf{R})$ such that*:

(i) *h is a linear isometry with respect to d_∞ and $\|\cdot\|_\infty$*,
(ii) *h is a homomorphism of sup-semilattices* (*and so is positive*),
(iii) $h(I_B) = 1$,
(iv) *$h(\mathscr{F}_c) - h(\mathscr{F}_c)$ is dense and order-dense in $\mathscr{C}(\Gamma, \mathbf{R})$.*

We can finally prove our result on Korovkin systems in the space $\mathscr{C}(\Omega, \mathscr{F})$.

Theorem 3. *Assume that Ω is compact metric. Let $\mathscr{H} \subset \mathscr{C}(\Omega, \mathbf{R}^+)$ be a Korovkin system for* **id** *in $\mathscr{C}(\Omega, \mathbf{R})$. Let $u \in \mathscr{F}_c$ be such that $aI_B \subset u$ for some $a > 0$. Let T be an $\mathscr{F}$-operator such that $T(u) = u$. Then, $(\mathscr{H}u) \cup \mathscr{F}$ is a Korovkin system for T in $\mathscr{C}(\Omega, \mathscr{F})$.*

Proof. By Lemma 2, $\mathscr{C}(\Omega, \mathscr{F}_c)$ embeds into the Banach lattice $\mathscr{C}(\Omega, \mathscr{C}(\Gamma, \mathbf{R}))$. Besides, one checks routinely that the embedding $f \mapsto f \cdot h(u)$ from $\mathscr{C}(\Omega, \mathbf{R})$ to $\mathscr{C}(\Omega, \mathscr{C}(\Gamma, \mathbf{R}))$ is linear and $|f \cdot h(u)| = |f| \cdot h(u)$, i.e. it is a vector lattice homomorphism. Notice, for the latter identity, that $h(u) \geqslant 0$ by the positivity of h and the fact that $I_{\{0\}} \subset u$.

By the universal Korovkin-type property [1, Theorem 3.2.1], $\mathscr{H}$ is then a Korovkin system for **id**, for mappings from $\mathscr{C}(\Omega, \mathbf{R})$ to $\mathscr{C}(\Omega, \mathscr{C}(\Gamma, \mathbf{R}))$.

Observe that $d_\infty(au, bu) = |a - b| \cdot \|u_0\|$ and $0 \leqslant a \leqslant b \Rightarrow au \subset bu$, so that the mapping $a \in \mathbf{R}^+ \mapsto a \cdot u$ is a monotone linear homeomorphism. Therefore $\mathscr{C}(\Omega, \mathbf{R}^+ \cdot u)$ and $\mathscr{C}(\Omega, \mathbf{R}^+)$ can be identified as far as we are concerned. We deduce that $\mathscr{H}u$ is a Korovkin system for **id**, for mappings from $\mathscr{C}(\Omega, \mathbf{R}^+ \cdot u)$ to $\mathscr{C}(\Omega, \mathscr{F}_c)$.

We are now ready to prove that $(\mathscr{H}u) \cup \mathscr{F}$ is a Korovkin system for T in $\mathscr{C}(\Omega, \mathscr{F})$. Assume that $T_n(fu) \to T(fu)$ and $T_n(A) \to T(A)$ for all $f \in \mathscr{H}$ and $A \in \mathscr{F}$. We have to prove that indeed $T_n(X) \to T(X)$ for all $X \in \mathscr{C}(\Omega, \mathscr{F})$.

Begin by noting that $T(fu) = fu$ for all $f \in \mathscr{H}$. Indeed, for any $y \in \Omega$,

$$T(f(y)u) = \lim_n T_n(f(y)u) = f(y) \lim_n T_n(u) = f(y)T(u) = f(y)u$$

by the linearity of T_n and the convergence for constant functions. Now T is an $\mathscr{F}$-operator, so that $T(fu)(y) = T(f(y)u)(y) = f(y)u$ for all $y \in \Omega$, i.e. $T(fu) = fu$.

Denote the metric in Ω by ρ. Since the mappings $\rho(x, \cdot)$ are continuous for all $x \in \Omega$, the Korovkin property of $\mathscr{H}u$ implies that

$$T_n(\rho(x, \cdot)u) \to \rho(x, \cdot)u$$

for each $x \in \Omega$. Here it is very important to realize that we have to prove that all $T_n(\rho(x, \cdot)u)$ take on values in $\mathscr{F}_c$, otherwise the last step would not be valid, as $\mathscr{H}u$ need not have the Korovkin property for general $\mathscr{F}$-valued mappings (see Example 1). In order to do so, notice that

$$T_n(\rho(x, \cdot)u) = T_n\left(\sum_{i=1}^{m} m^{-1}\rho(x, \cdot)u\right) = \sum_{i=1}^{m} (m^{-1}T_n(\rho(x, \cdot)u))$$

by the quasiconcavity of the values of $\rho(x,\cdot)u$ and the linearity of T_n. Now, by an application of the Artstein–Hansen convexification lemma [2] to each upper level set,

$$\sum_{i=1}^{m}(m^{-1}T_n(\rho(x,\cdot)u)(y))\to_m \text{qco}\, T_n(\rho(x,\cdot)u)(y)$$

for each $y\in\Omega$. That is, qco $T_n(\rho(x,\cdot)u)=T_n(\rho(x,\cdot)u)$. (One just needs the existence of the limit in *some* Hausdorff topology, but let us be precise and say that it is so in the weak topology generated by the mappings $A\mapsto A_\alpha$ for $\alpha\in(0,1]$.)

Finally, ρ fulfils the assumptions on ϕ in the statement of Theorem 1, we have shown that

$$T_n(\rho(x,\cdot)u)\to\rho(x,\cdot)u=T(\rho(x,\cdot)u)$$

and by hypothesis $T_n(A)\to T(A)$ for all $A\in\mathscr{F}$. It follows then, from implication (ii) $\Rightarrow$ (i), that $T_n(X)\to T(X)$ for all $X\in\mathscr{C}(\Omega,\mathscr{F})$. The proof is complete. □

Notice that it is not really necessary to assume that Ω is metric: only the existence of a mapping ϕ as in the statement of Theorem 1 is needed. Besides, the hypothesis $T(u)=u$ can clearly be slightly relaxed to $Tu=\lambda u$ for some $\lambda\geqslant 0$.

Pushing the same ideas just a bit further, one can ascertain the connection between Korovkin systems in $\mathscr{C}(\Omega,\mathscr{F}_c)$ and Korovkin systems in $\mathscr{C}(\Omega,\mathscr{F})$.

Theorem 4. *Let $\mathscr{J}$ be a Korovkin system for* **id** *in $\mathscr{C}(\Omega,\mathscr{F}_c)$. Then, $\mathscr{J}\cup\mathscr{F}$ is so in $\mathscr{C}(\Omega,\mathscr{F})$. Conversely, if $\mathscr{L}$ is a Korovkin system for* **id** *in $\mathscr{C}(\Omega,\mathscr{F})$, then $\mathscr{L}'=\{\text{qco}\,X\mid X\in\mathscr{L}\}$ is so in $\mathscr{C}(\Omega,\mathscr{F}_c)$.*

Proof. The proof of the first claim is easy. For if $\mathscr{J}$ has the Korovkin property in $\mathscr{C}(\Omega,\mathscr{F}_c)$, convergence of a sequence of linear positive operators in $\mathscr{J}$ implies convergence in $\mathscr{H}u$, where $\mathscr{H}$ and u are as in Theorem 3. But $(\mathscr{H}u)\cup\mathscr{F}$ being a Korovkin system in $\mathscr{C}(\Omega,\mathscr{F})$, it is clear that $\mathscr{J}\cup\mathscr{F}$ is so also.

As to the second claim, we know from the proof of Theorem 3 that a linear operator from $\mathscr{C}(\Omega,\mathscr{F}_c)$ to $\mathscr{C}(\Omega,\mathscr{F})$ actually takes on values in $\mathscr{C}(\Omega,\mathscr{F}_c)$. Then, qco X is just the restriction of X to $\mathscr{C}(\Omega,\mathscr{F}_c)$. It follows that $\mathscr{L}'$ has the Korovkin property. □

Example 1. It is easy to show that $\mathscr{J}$ alone does not have the Korovkin property in the whole of $\mathscr{C}(\Omega,\mathscr{F})$. We can even take $\mathscr{J}=\mathscr{C}(\Omega,\mathscr{F}_c)$. Let $T_n=\text{qco}$ for all $n\in\mathbf{N}$ and $T=\mathbf{id}$. But $T_n(X)\nrightarrow T(X)$ for any $X\notin\mathscr{C}(\Omega,\mathscr{F}_c)$.

Theorems 3 and 4 can be particularized to functions whose values are (possibly non-convex) sets.

Corollary 5. *Assume that Ω is compact metric. Let $\mathscr{H}\subset\mathscr{C}(\Omega,\mathbf{R}^+)$ be a Korovkin system for* **id** *in $\mathscr{C}(\Omega,\mathbf{R})$. Let $L\in\mathscr{K}_c$ be such that $aB\subset L$ for some $a>0$. Let S be a $\mathscr{K}$-operator such that $S(L)=L$. Then, $(\mathscr{H}L)\cup\mathscr{K}$ is a Korovkin system for S in $\mathscr{C}(\Omega,\mathscr{K})$.*

Besides, if $\mathscr{J}$ *is a Korovkin system for* **id** *in* $\mathscr{C}(\Omega, \mathscr{K}_c)$, *then* $\mathscr{J} \cup \mathscr{K}$ *is so in* $\mathscr{C}(\Omega, \mathscr{K})$. *Conversely, if* $\mathscr{L}$ *is a Korovkin system for* **id** *in* $\mathscr{C}(\Omega, \mathscr{K})$, *then* $\mathscr{L}' = \{\text{co}\, X \mid X \in \mathscr{L}\}$ *is so in* $\mathscr{C}(\Omega, \mathscr{K}_c)$.

The proof uses the identification $K \in \mathscr{K} \leftrightarrow I_K \in \mathscr{F}$.

In [18], we have given a method to show that Korovkin systems for d_∞ convergence are also Korovkin systems for some weaker types of convergence. This can be used together with the ideas in this paper to extend Theorems 3 and 4 in a similar way. Details are left to the reader.

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